EVERYDAY EXAMPLES OF ENGINEERING CONCEPTS

D10: Kinematics of rigid bodies in three dimensions

This is an extract from 'Real Life Examples in Dynamics: Lesson plans and solutions' edited by Eann A. Patterson, first published in 2006 (ISBN:978-0-615-20394-2) which can be obtained on-line at www.engineeringexamples.org and contains suggested exemplars within lesson plans for Sophomore Solids Courses. Prepared as part of the NSF-supported project (#0431756) entitled: "Enhancing Diversity in the Undergraduate Mechanical Engineering Population through Curriculum Change".

INTRODUCTION

(from *'Real Life Examples in Dynamics: Lesson plans and solutions'*)

These notes are designed to enhance the teaching of a junior level course in dynamics, increase the accessibility of the principles, and raise the appeal of the subject to students from diverse backgrounds. The notes have been prepared as skeletal lesson plans using the principle of the 5Es: Engage, Explore, Explain, Elaborate and Evaluate. The 5E outline is not original and was developed by the Biological Sciences Curriculum Study[1] in the 1980s from work by Atkin and Karplus[2] in 1962. Today this approach is considered to form part of the constructivist learning theory and a number of websites provide easy-to-follow explanations of them[3].

These notes are intended to be used by instructors and are written in a style that addresses the instructor, however this is not intended to exclude students who should find the notes and examples interesting, stimulating and hopefully illuminating, particularly when their instructor is not utilizing them. In the interest of brevity and clarity of presentation, standard derivations and definitions are not included since these are readily available in textbooks which these notes are not intended to replace but rather to supplement and enhance. Similarly, it is anticipated that these lessons plans can be used to generate lectures/lessons that supplement those covering the fundamentals of each topic.

It is assumed that students have acquired a knowledge and understanding of topics usually found in a Sophomore level course in Statics, including free-body diagrams and efficiency.

This is the second in a series of such notes. The first in the series entitled 'Real Life Examples in Mechanics of Solids' edited by Eann Patterson (ISBN: 978-0-615-20394-2) was produced in 2006 and is available on-line at www.engineeringexamples.org.

Acknowledgements

Many of these examples have arisen through lively discussion in the consortium supported by the NSF grant (#0431756) on "Enhancing Diversity in the Undergraduate Mechanical Engineering Population through Curriculum Change" and the input of these colleagues is cheerfully acknowledged as is the support of National Science Foundation.

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[1] Engleman, Laura (ed.), *The BSCS Story: A History of the Biological Sciences Curriculum Study*. Colorado Springs: BSCS, 2001.
[2] Atkin, J. M. and Karplus, R. (1962). Discovery or invention? *Science Teacher* 29(5): 45.
[3] e.g. Trowbridge, L.W., Bybee, R.W., *Becoming a secondary school science teacher*. Merrill Pub. Co. Inc., 1990.

THREE DIMENSIONAL RIGID BODY MOTION

10. Topic: **Kinematics of rigid bodies in three dimensions**

Engage:

If you play the violin, then bring it to class and play an excerpt from your favorite piece; or if you don't play, maybe you could ask a student in your class or a colleague in the music school. And, or search in www.YouTube.com for "Janine Jansen Mozart" and select Mozart Violin Concerto 5 (2of 5) for a good close-up of the violin being played[4]. Play it a second time and ask the students play along on the "air violin".

Explore:

Again in www.YouTube.com search under 'Robot violinist' and show the video of the ToyotaViolin playing robot[5]. Discuss the need to calculate displacements, rotations, velocities, accelerations, forces and moments in order to be able to program the robot to perform such a complex task. It would be relevant to introduce examples of engineering applications of robots, e.g. welding robots[6]; again search in www.YouTube.com for suitable videos (also see "Robot of the Year 2007"[7]).

Ask students to explore in pairs the degrees of freedom and axes of rotation in their own arms. Ask them to identify the degrees of freedom used in playing the "air violin"

Explain:

You have seven degrees of freedom in your arm: at the shoulder, elbow and wrist, as shown in the picture. Use your own arm, or ask a student to help you. If you have brought a violin to class then talk about the motions involved in moving the bow to the instrument.

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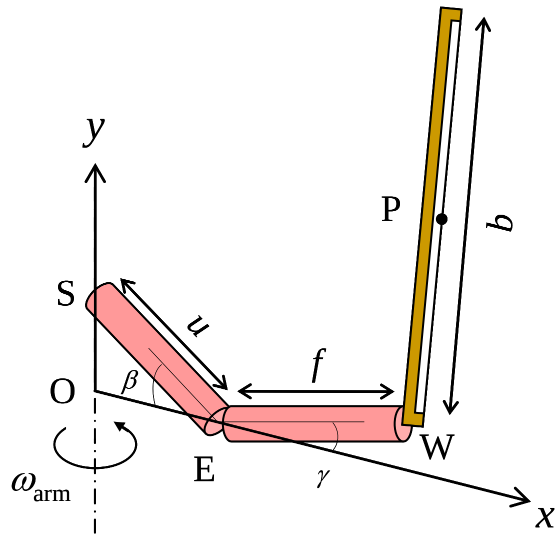
4 http://www.youtube.com/watch?v=5QlyMNMLlfI
5 http://www.youtube.com/watch?v=EzjkBwZtxp4
6 http://www.youtube.com/watch?v=362vMN7Ra4w
7 http://www.youtube.com/watch?v=Q8WnAN9jmEc

Elaborate

A simple robotic arm for a violin playing robot is shown in the figure. To engage the bow with the instrument, the arm will rotate about the vertical axis shown with the elbow at E fixed. To avoid damage to the bow and strings we need to calculate the angular velocity and acceleration of the bow.



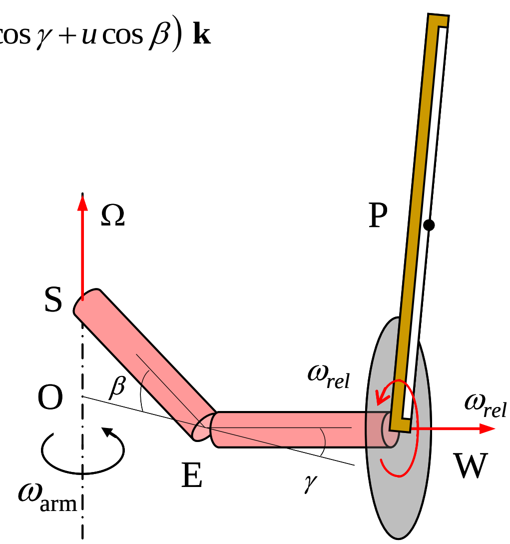
We can establish a coordinate system which is fixed relative to the arm with the *x*-axis in the plane SEW and the *y*-axis co-incident with the axis of rotation, i.e with an origin at O. So the angular velocity of the arm, ω and of the coordinate system, Ω are equal and $\omega_{arm} = \mathbf{\Omega} = \omega_0 \mathbf{j}$. Thus, we can consider the position vector of W relative to O:

$$\mathbf{r}_{W/O} = \left(f\cos\gamma + u\cos\beta\right)\mathbf{i} - \left(f\sin\gamma + u\sin\beta\right)\mathbf{j}$$

now, "$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega}\times\mathbf{r}_{A/B}$"

so $$\mathbf{v}_W = \mathbf{v}_O + \boldsymbol{\omega}_{arm}\times\mathbf{r}_{W/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_0 & 0 \\ f\cos\gamma + u\cos\beta & -f\sin\gamma - u\sin\beta & 0 \end{vmatrix}$$

and $$\mathbf{v}_W = -\omega_0\left(f\cos\gamma + u\cos\beta\right)\mathbf{k}$$



The coordinate system is fixed with respect to the arm so the angular velocity of the bow with respect to the coordinate system is:

$$\boldsymbol{\omega}_{rel} = \omega_{rel}(\cos\gamma)\ \mathbf{i} - \omega_{rel}(\sin\gamma)\ \mathbf{j}$$

and the position of P, halfway along the bow, relative to W is:

$$\mathbf{r}_{P/W} = -(b\sin\gamma)\ \mathbf{i} - (b\cos\gamma)\ \mathbf{j}$$

so, the velocity of the point, P is given by:

$$\mathbf{v}_P = \mathbf{v}_W + \boldsymbol{\omega}_{bow}\times\mathbf{r}_{P/w} = -\omega_0\left(f\cos\gamma + u\cos\beta\right)\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_{rel}\cos\gamma & \omega_0 - \omega_{rel}\sin\gamma & 0 \\ -\frac{b}{2}\sin\gamma & -\frac{b}{2}\cos\gamma & 0 \end{vmatrix}$$

$$\mathbf{v}_P = \left[-\omega_0\left(f\cos\gamma + u\cos\beta\right) + \left(\omega_{rel}\cos\gamma - \frac{b}{2}\cos\gamma\right) - \left(\omega_0 - \omega_{rel}\sin\gamma - \frac{b}{2}\sin\beta\right)\right]\mathbf{k}$$

For a bow of length 75cm held by an arm with upper and forearm lengths of 25cm and 30cm respectively at angles of $\beta = 15°$ and $\gamma = 20°$, when the arm rotates at 0.75 rad/s ($=\omega_0$) and the wrist rotates the bow at 0.5rad/s ($=\omega_{rel}$) the velocity of point, P is given by:

$$\mathbf{v}_P = \left[-0.75\times(0.3\cos 20 + 0.25\cos 15) + (0.5\cos 20 - 0.375\cos 20) - (0.75 - 0.5\sin 20 - 0.375\sin 15)\right]$$
$$\mathbf{v}_P = \left[-0.393 - 0.117 - 0.482\right] = 0.99\,\text{m/s}$$

The acceleration of the bow, $\boldsymbol{\alpha}$ is given by:

$$\boldsymbol{\alpha} = \frac{d\omega_x}{dt}\mathbf{i} + \frac{d\omega_y}{dt}\mathbf{j} + \frac{d\omega_z}{dt}\mathbf{k} + \boldsymbol{\Omega}\times\boldsymbol{\omega}$$

though the components of ω are constant, the acceleration is not zero but

$$\boldsymbol{\alpha}_{bow} = \boldsymbol{\Omega}\times\boldsymbol{\omega}_{bow} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_{arm} & 0 \\ \omega_{rel}\cos\gamma & \omega_{arm} - \omega_{rel}\sin\gamma & 0 \end{vmatrix}$$

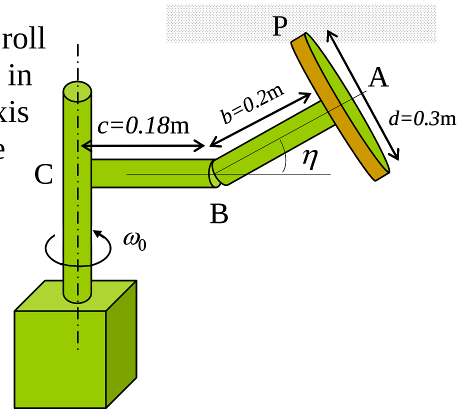
$$= \omega_{arm}\omega_{rel}\cos\gamma\,\mathbf{k}$$

$$= -0.75\times 0.5\times\cos 20 = 0.35\,\text{rad/s about the } z\text{-axis.}$$

Evaluate

Example 10.1

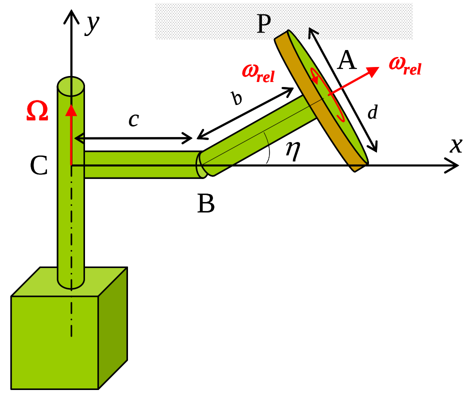
An industrial robot uses its arm in the position shown to roll out adhesive tape on the underside of a structure as shown in the figure. If the arm CB rotates about the vertical axis through C at a constant angular velocity of 3 rad/s find the velocity at which the tape is being laid out when $\eta = 35°$.



Solution:

Establish a coordinate system that is fixed relative to the arm with the *x*-axis coincident with CB and the *y*-axis co-incident with the axis of rotation, i.e with an origin at C. Then

$$\omega_{arm} = \Omega = \omega_0 \mathbf{j}$$



and we can consider the position vector of A relative to C:

$$\mathbf{r}_{A/C} = \left(c + b\cos\eta\right)\,\mathbf{i} - (c\sin\eta)\mathbf{j}$$

now $\mathbf{v}_A = \mathbf{v}_C + \boldsymbol{\omega}_{arm} \times \mathbf{r}_{A/C} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_0 & 0 \\ c + b\cos\eta & -c\sin\eta & 0 \end{vmatrix}$

and $\mathbf{v}_A = -\omega_0\left(c + b\cos\eta\right)\,\mathbf{k}$

The coordinate system is fixed with respect to the arm so the angular velocity of the disc with respect to the coordinate system is:

$$\boldsymbol{\omega}_{rel} = \omega_{rel}\left(\cos\eta\right)\,\mathbf{i} - \omega_{rel}\left(\sin\eta\right)\,\mathbf{j}$$

For the point of contact, P there is no motion due to the adhesive on the tape so $\mathbf{v}_P = 0$. Now, the position of P relative to A is:

$$\mathbf{r}_{P/A} = -\left(\frac{d}{2}\sin\eta\right)\mathbf{i} - \left(\frac{d}{2}\cos\eta\right)\mathbf{j}$$

So, the velocity of the point, P is given by:

$$\mathbf{v}_P = \mathbf{v}_A + \boldsymbol{\omega}_{disk} \times \mathbf{r}_{P/A} = -\omega_0\left(c + b\cos\eta\right)\,\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_{rel}\cos\eta & \omega_0 - \omega_{rel}\sin\eta & 0 \\ -\frac{d}{2}\sin\eta & -\frac{d}{2}\cos\eta & 0 \end{vmatrix} = 0$$

hence, $\omega_{rel} = -\omega_0\left(\frac{2c}{d} + \frac{2b}{d}\cos\eta - \sin\eta\right)$

and $\boldsymbol{\omega}_{disk} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{rel} = \omega_{rel}\cos\eta\ \mathbf{i} + (\omega_0 - \omega_{rel}\sin\eta)\mathbf{j}$

finally, the tape is used at

$$\mathbf{v}_T = \boldsymbol{\omega}_{disc} \times \mathbf{r}_{P/A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_{rel}\cos\eta & \omega_0 - \omega_{rel}\sin\eta & 0 \\ -\frac{d}{2}\sin\eta & -\frac{d}{2}\cos\eta & 0 \end{vmatrix}$$

$$\mathbf{v}_T = \omega_0(c + b\cos\eta)\ \mathbf{k} = 3\times(0.18 + 0.2\cos 35) = 1.03\,\text{m/s}$$

So the tape is used at approximately 1 meter per second.

Example 10.2

What degrees of freedom are involved in cleaning your teeth using a tooth brush? Calculate the approximate velocities of the parts of your arm during a brushing stroke.

Note:

The determinant of a 3x3 matrix, $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

is given by $\det[\mathbf{A}] = aei - afh + bfg - bdi + cdh - ceg$

and can remember using the diagonals as follows: