EVERYDAY EXAMPLES OF ENGINEERING CONCEPTS

F7: Similitude & dimensional analysis

This is an extract from 'Real Life Examples in Fluid Mechanics: Lesson plans and solutions' edited by Eann A. Patterson, first published in 2011 (ISBN:978-0-9842142-3-5) which can be obtained on-line at www.engineeringexamples.org and contains suggested exemplars within lesson plans for Sophomore Fluids Courses. They were prepared as part of the NSF-supported project (#0431756) entitled: "Enhancing Diversity in the Undergraduate Mechanical Engineering Population through Curriculum Change".

INTRODUCTION

(from *'Real Life Examples in Fluid Mechanics: Lesson plans and solutions'*)

These notes are designed to enhance the teaching of a sophomore level course in fluid mechanics, increase the accessibility of the principles, and raise the appeal of the subject to students from diverse backgrounds. The notes have been prepared as skeletal lesson plans using the principle of the 5Es: Engage, Explore, Explain, Elaborate and Evaluate. The 5E outline is not original and was developed by the Biological Sciences Curriculum Study[1] in the 1980s from work by Atkin & Karplus[2] in 1962. Today this approach is considered to form part of the constructivist learning theory[3].

These notes are intended to be used by instructors and are written in a style that addresses the instructor, however this is not intended to exclude students who should find the notes and examples interesting, stimulating and hopefully illuminating, particularly when their instructor is not utilizing them. In the interest of brevity and clarity of presentation, standard derivations, common tables/charts, and definitions are not included since these are readily available in textbooks which these notes are not intended to replace but rather to supplement and enhance. Similarly, it is anticipated that these lesson plans can be used to generate lectures/lessons that supplement those covering the fundamentals of each topic.

It is assumed that students have acquired a knowledge and understanding the following topics: first and second law of thermodynamics, Newton's laws, free-body diagrams, and stresses in pressure vessels.

This is the fourth in a series of such notes. The others are entitled 'Real Life Examples in Mechanics of Solids', 'Real Life Examples in Dynamics' and 'Real Life Examples in Thermodynamics'. They are available on-line at www.engineeringexamples.org.

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[1] Engleman, Laura (ed.), *The BSCS Story: A History of the Biological Sciences Curriculum Study*. Colorado Springs: BSCS, 2001.
[2] Atkin, J. M. and Karplus, R. (1962). Discovery or invention? *Science Teacher* 29(5): 45.
[3] e.g. Trowbridge, L.W., and Bybee, R.W., *Becoming a secondary school science teacher*. Merrill Pub. Co. Inc., 1990.

MODELLING

7. Topic**: Similitude and dimensional analysis**

Engage:

Take your toy boats from the bath into class. If you don't have any, then you could either borrow or buy some; or at the end of the previous class invite students to bring in their own bath toys.

Show them a video from YouTube of a ship in rough seas (search using '*Abeille Flandre*' and show the clip of this name[4].

Explore:

We have all played with boats in the bath. Discuss whether the behavior of toy boats will provide a good model for predicting the behavior of full-scale ships at sea. We could get out of the bath and conduct the experiment in the controlled environment of the laboratory; but would the behavior in the laboratory be a good prediction of the performance on the high seas?

Ask students to construct a list of factors that could differ between the lab and the ocean and then construct a master list on the board.

Explain:

You should have ended up with a long list of factors that could vary between the lab and the full-scale performance at sea. Explain that it is advantageous to group variables such as pressure, density, length, viscosity, velocity into non-dimensional groups and then to conduct experiments to establish the functional relationship between the groups rather than the variables. This considerably reduces the amount of experimentation required and can also help in ensuring similitude between experiments and the prototype or full-scale version.

Elaborate:

An American physicist, Edgar Buckingham (1867-1940), showed that the number of non-dimensional groups required to correlate the variables in a certain process is given by n-m where n is the number of variables to be grouped and m is the number of basic dimensions included amongst the variables. So we might expect the force, F, acting on our ship, and toy boat, would be a function of fluid density, ρ; dynamic viscosity, μ; gravity, g; the speed of the ship, v; and a characteristic dimension of the ship, l.

So, $F = f(\rho, \mu, g, v, l)$

[4] www.youtube.com/watch?v=3408T5A-ApU

The fundamental dimensions of these quantities are

F	N (= kg m s-2)	MLT^{-2}
ρ	kg m^{-3}	ML^{-3}
μ	N s m^{-2}	$ML^{-1}T^{-1}$
g	m s^{-2}	LT^{-2}
v	m s^{-1}	LT^{-1}
l	m	L

So we have six variables and three basic dimensions (M: mass, L: length, and T: time) and thus we will need three $(=6-3)$ non-dimensional or Pi groups (hence the name Buckingham-Pi approach). This means we can reduce the number of variables from six to three.

To find the first group, Π_1, we can select the dependent variable, F, and form a non-dimensional group by introducing variables with the appropriate dimensions in such a way as to create the non-dimensional group. If a dimension exists on its own as the only dimension of a variable, then it should be considered last. So in this case starting with [M] we can introduce ρ and then v^2 to have $[T^{-2}]$ but then we have $[L^{-3}][L^2] = [L^{-1}]$ and we need [L] so we must introduce l^2. So

$$\Pi_1 = \frac{F}{\rho v^2 l^2}$$

This is effectively the ratio of the shear force on the hull to the inertia forces. The process can be repeated choosing μ to form the second group

$$\Pi_2 = \frac{\mu}{vl\rho}$$

Note that the same repeating variables are used to achieve the non-dimensional group. This group is the Reynolds number and describes the ratio of the inertia to shear forces in the fluid. And finally for third group using g we can obtain

$$\Pi_3 = \frac{gl}{v^2}$$

This is the Froude number and describes the ratio of inertia to gravitational forces for a fluid with a free surface. So the functional relationship for the behavior of the ship either as a model (toy boat) or full-scale in the ocean is

$$\frac{F}{\rho v^2 l^2} = \phi\left(\frac{\mu}{vl\rho}, \frac{gl}{v^2}\right)$$

In order to achieve similitude between a model and a prototype Pi groups are made equivalent. So, for instance, if a tug boat in your bath is 6cm long and an ocean-going tug is 32m long then Π_3 above would imply that

$$\frac{gl_{bath}}{v_{bath}^2} = \frac{gl_{ocean}}{v_{ocean}^2} \text{ or } \frac{v_{bath}^2}{v_{ocean}^2} = \frac{l_{bath}}{l_{ocean}} = \frac{0.6}{32} = \frac{1}{533}$$

And the speed of your model needs to be 1/23 ($=1/\sqrt{533}$) of the speed of the ocean-going tug, which is just as well since a typical speed for the ocean version is 11 knots or 5.6 m/s. The model would have to move at 0.24 m/s (=5.6/23), or about 9 inches/sec, which is quite fast for the bath.

Evaluate:

Ask the students to attempt the following examples:

Example 7.1

The suction of a vacuum cleaner can be equated with the pressure drop across its fan, Δp which is in turn related the fan diameter, D; its axial length, l; the rotational speed, ω; the inlet and outlet diameters, d_1 and d_2 and the air density, ρ. Find the functional relationship between these groups.

Solution:

$$\Delta p = f\left(D, l, \omega, d_1, d_2, \rho\right)$$

We can express the dimensions of the variables as follows:

Δp	Pa (N m^{-2})	ML^{-1}T^{-2}
D	m	L
l	m	L
ω	rad s^{-1}	T^{-1}
d_1	m	L
d_2	m	L
ρ	kg m^{-3}	ML^{-3}

There are seven variables and three basic dimensions so there will be four non-dimensional groups. The repeating variables are D, ω and ρ. The first group can be formed around Δp and we obtain

$$\Pi_1 = \frac{\Delta p}{\rho \omega^2 D^2}$$

For the second group use the next non-repeating variable and so on to give

$$\Pi_2 = \frac{l}{D},\ \Pi_3 = \frac{d_1}{D} \text{ and } \Pi_4 = \frac{d_2}{D}$$

and $$\frac{\Delta p}{\rho \omega^2 D^2} = \phi\left(\frac{l}{D}, \frac{d_1}{D}, \frac{d_2}{D}\right)$$

Example 7.2

In order to study the interaction of a micro-surgery device and the flow in an artery, a five times scale model of an artery is to be constructed. The volume flow rate, Q, in the artery is believed to be a function of frequency of the heart beat, ω; artery diameter, D; the fluid density, ρ;

viscosity, μ; and the pressure gradient, $\Delta p/\Delta l$. Identify the dimensional groups and estimate the volume flow rate required if saline is used as the work fluid instead of blood.

Solution:

$$Q = \phi\left(\omega, D, \rho, \mu, \frac{\Delta p}{\Delta l}\right)$$

We can express the dimensions of the variables as follows:

Q	m^3s^{-1}	L^3T^{-1}
ω	s^{-1}	T^{-1}
D	m	L
ρ	kg m^{-3}	ML^{-3}
μ	N s m^{-2}	$ML^{-1}T^{-1}$
$\Delta p/\Delta l$	Pa/m (N m^{-3})	$ML^{-2}T^{-2}$

There are six variables and three basic dimensions so there will be three non-dimensional groups. The repeating variables are *f*, *D* and ρ. The first group can be formed around Δp and we obtain

$$\Pi_1 = \frac{Q}{D^3\omega}$$

Then taking each of the non-repeating variables in turn

$$\Pi_2 = \frac{\mu}{\rho D^2\omega} \quad \text{and} \quad \Pi_3 = \frac{\Delta p/\Delta l}{\rho D\omega^2}$$

So, if the working fluid is changed from blood to saline, Π_2 equivalence must be maintained so

$$\omega_s = \frac{\mu_s}{\mu_b}\frac{\rho_b D_b^2}{\rho_s D_s^2}\omega_b = \frac{2\times10^{-3}}{4\times10^{-3}}\cdot\frac{1060\times1}{1200\times2^2}\omega_b = 0.11\omega_b$$

assuming the saline is formulated to give approximately the same viscosity as blood, the model heart beat needs to be about 1/9th of the natural heart rate. Now for flowrate equivalence of the first Pi group is required so

$$Q_s = \frac{D_s^3\omega_s}{D_b^3\omega_b}Q_b = \frac{2^3}{1}\frac{1}{0.044}Q_b = 72Q_b$$

Hence, the volume flowrate would need to be about 70 times the natural value in the body.