EVERYDAY EXAMPLES OF ENGINEERING CONCEPTS

D12: Free & forced vibrations

This is an extract from 'Real Life Examples in Dynamics: Lesson plans and solutions' edited by Eann A. Patterson, first published in 2006 (ISBN:978-0-615-20394-2) which can be obtained on-line at www.engineeringexamples.org and contains suggested exemplars within lesson plans for Sophomore Solids Courses. Prepared as part of the NSF-supported project (#0431756) entitled: "Enhancing Diversity in the Undergraduate Mechanical Engineering Population through Curriculum Change".

INTRODUCTION

(from *'Real Life Examples in Dynamics: Lesson plans and solutions'*)

These notes are designed to enhance the teaching of a junior level course in dynamics, increase the accessibility of the principles, and raise the appeal of the subject to students from diverse backgrounds. The notes have been prepared as skeletal lesson plans using the principle of the 5Es: Engage, Explore, Explain, Elaborate and Evaluate. The 5E outline is not original and was developed by the Biological Sciences Curriculum Study[1] in the 1980s from work by Atkin and Karplus[2] in 1962. Today this approach is considered to form part of the constructivist learning theory and a number of websites provide easy-to-follow explanations of them[3].

These notes are intended to be used by instructors and are written in a style that addresses the instructor, however this is not intended to exclude students who should find the notes and examples interesting, stimulating and hopefully illuminating, particularly when their instructor is not utilizing them. In the interest of brevity and clarity of presentation, standard derivations and definitions are not included since these are readily available in textbooks which these notes are not intended to replace but rather to supplement and enhance. Similarly, it is anticipated that these lessons plans can be used to generate lectures/lessons that supplement those covering the fundamentals of each topic.

It is assumed that students have acquired a knowledge and understanding of topics usually found in a Sophomore level course in Statics, including free-body diagrams and efficiency.

This is the second in a series of such notes. The first in the series entitled 'Real Life Examples in Mechanics of Solids' edited by Eann Patterson (ISBN: 978-0-615-20394-2) was produced in 2006 and is available on-line at www.engineeringexamples.org.

Acknowledgements

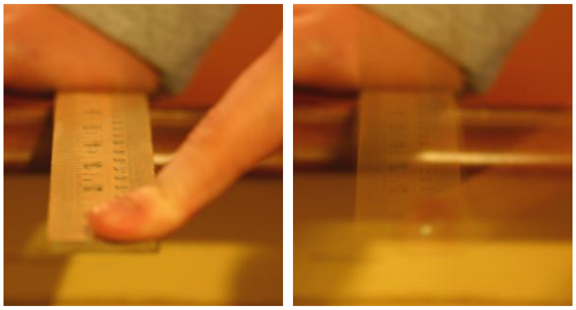
Many of these examples have arisen through lively discussion in the consortium supported by the NSF grant (#0431756) on "Enhancing Diversity in the Undergraduate Mechanical Engineering Population through Curriculum Change" and the input of these colleagues is cheerfully acknowledged as is the support of National Science Foundation.

Eann A. Patterson
A.A. Griffith Chair of Structural Materials and Mechanics
School of Engineering, University of Liverpool, Liverpool, UK
& Royal Society Wolfson Research Merit Award Recipient

[1] Engleman, Laura (ed.), *The BSCS Story: A History of the Biological Sciences Curriculum Study*. Colorado Springs: BSCS, 2001.
[2] Atkin, J. M. and Karplus, R. (1962). Discovery or invention? *Science Teacher* 29(5): 45.
[3] e.g. Trowbridge, L.W., Bybee, R.W., *Becoming a secondary school science teacher*. Merrill Pub. Co. Inc., 1990.

MECHANICAL VIBRATIONS

12. Topic**: Free and forced vibrations**



Engage:

Take a hula hoop and a wooden ruler into class. Place the ruler overhanging the bench, lean on the end on the bench and flick the free end so that the ruler vibrates. Repeat this a few times and slide the ruler onto the bench, as it vibrates, so that the pitch of the noise changes – the frequency will go up.

Suggest that the students do the experiment for themselves.

This is noisy experiment so good for engaging the students but the mathematics is quite hard so tell them this and put it to one side. Pick up your hula hoop and, if you can, do some hooping (see www.hooping.org for instructions on hula hooping and links on how to make your own).

Explore:

Allow the hoop to oscillate on your finger, in simple harmonic motion, and discuss the characteristics of the motion.

The moment of inertia of a thin ring or hoop about an axis through its center is $I_C = mr^2$

and using the parallel axis system the moment of inertia about a point, E on the hoop is $I_E = I_C + mr^2 = 2mr^2$

The kinetic energy of a hoop oscillating on your finger is the sum of its translational (=zero) and rotational kinetic energy:

$$KE = \frac{mv_E^2}{2} + \frac{I_E \omega_n^2}{2} = mr^2\dot{\theta}^2$$

The potential energy is the sum of the elastic (=zero) and gravitational potential energies. For small oscillations the center of gravity of the hoop is raised by $r(1-\cos\theta)$ and for small angles we can use the series expansion of $\cos\theta = 1 - \left(\theta^2/2\right)$ so:

$$PE = mgh = mgr\frac{\theta^2}{2}$$

Now, the total energy of the system is constant:

$$KE + PE = mr^2\dot{\theta}^2 + mgr\frac{\theta^2}{2} = \text{constant}$$

And its derivative yields the equation of motion of the hoop, i.e.

$$mr^2 2\dot{\theta}\ddot{\theta} + mgr\theta\dot{\theta} = 0$$

or $$mr\dot{\theta}\left(2r\ddot{\theta} + g\theta\right) = 0$$

and $\dot{\theta}$ is not always zero, hence:

$$\ddot{\theta} + \frac{g}{2r}\theta = 0$$

This is the equation of motion for the undamped free vibration of a simple block and spring system which could be used to represent the oscillating hoop. It is known as simple harmonic motion in which the angular acceleration is proportional to the angular displacement of the hoop. The natural frequency of oscillation is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{2r}}$$

If you oscillate the hoop on your finger at this frequency you will generate the largest oscillations. For a 1m diameter hoop this frequency is about ½ Hz $\left(= \sqrt{9.81}/2\pi\right)$.

The solution to the equation of motion is:

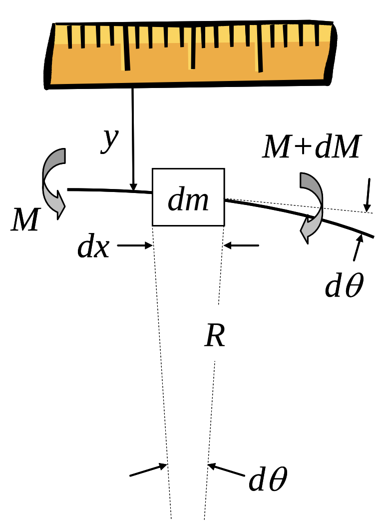
$$x(t) = A\sin\sqrt{\frac{g}{2r}}t + B\cos\sqrt{\frac{g}{2r}}t$$

where A and B must determined from known boundary conditions. The amplitude of the oscillation is $\sqrt{A^2 + B^2}$.

Explain:

Now, return to the ruler and repeat the analysis. The kinetic energy stored in the ruler is

$$"KE = \frac{mv^2}{2}"$$



and for an element this is

$$d(KE) = \frac{\dot{y}^2 dm}{2}$$

So to obtain the total strain energy in the ruler we need to integrate over its length, i.e.

$$KE = \frac{1}{2}\int_0^l \dot{y}^2 dm$$

And in one period, τ, any point on the ruler moves through $\hat{y}$ so

$$\dot{y} = \frac{\hat{y}}{\tau} = \omega\,\hat{y}$$

and

$$KE = \frac{m\omega^2}{2}\int_0^l \hat{y}^2 dx$$

where *m* is the mass per unit length. Now considering the strain energy stored in terms of the bending moment, *M* and the curvature of the beam:

$$U = \frac{1}{2}\int M d\theta$$

and for small deflections:

$$\theta = \frac{dy}{dx} \qquad \text{and} \qquad \frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

also from the theory of beams:

$$\frac{1}{R} = \frac{M}{EI}$$

thus $U = \frac{1}{2}\int M d\theta = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx$

Neglecting the gravitational potential energy, the total energy of the ruler is:

$$KE + U = \frac{m\omega^2}{2}\int_0^l \hat{y}^2 dx + \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx = \text{constant}$$

Elaborate

From Mechanics of Solids, the deflection of a cantilever can be described as:

$$y = \frac{Pl^3}{6EI}\left[3\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right]$$

so the total energy can be demonstrated to be:

$$\left(\frac{Pl^3}{3EI}\right)^2\left(\frac{\omega^2}{2}\frac{33ml}{140} + \frac{3EI}{2l^3}\right) = \text{constant}$$

At the natural frequency we can equate the energies, so:

$$\frac{\omega^2}{2}\frac{33ml}{140} = \frac{3EI}{2l^3} \text{ and } \omega = \sqrt{\frac{140EI}{11ml^4}}$$

Thus as the ruler gets shorter the frequency goes up. For a wooden mahogany ruler (*E*=9,200 MPa) of cross-section 28mm×3mm and mass per unit length of 0.0459 kg/m (=14×10^{-3}/0.305):

$$I = \frac{bh^3}{12} = \frac{0.028\times 0.003^3}{12} = 6.3\times 10^{-11}\text{m}^4$$

At l=22cm $\omega = \sqrt{\dfrac{140EI}{11ml^4}} = \sqrt{\dfrac{140 \times 9.2 \times 10^9 \times 6.3 \times 10^{-11}}{11 \times 0.0459 \times 0.22^4}} = 261.63\,\text{Hz}$ i.e. middle C.

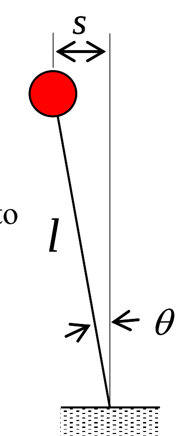
Whereas a plastic ruler (polycarbonate, E = 2.2 GPa) of cross-section 45×2mm (I =3×10^{-11} m^4) and mass per unit length 0.1076 kg/m requires a length of 10.3cm to produce middle C.

Evaluate

Ask students to complete the following examples.

Example 12.1

Find the natural frequency and the equation of motion for a ball of mass 12g stuck on the end of a whip antenna of length 0.7m.



Solution

For the ball, apply Newton's Second Law in the direction of motion, i.e. tangential to the support, F from the aerial:

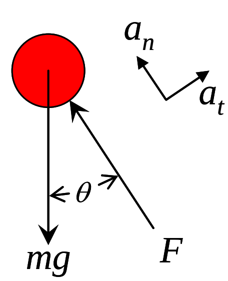
$$\sum F_t = ma_t \quad \text{so} \qquad -mg\sin\theta = ma_t$$

using kinematics, $$a_t = \frac{d^2x}{dt^2} = \ddot{x}$$

and from the geometry, for small angles,

$$x = l\theta$$

so we can rewrite the equation of motion as



$$-mg\sin\theta = ml\ddot{\theta} \quad \text{or} \quad \ddot{\theta} + \frac{g}{l}\theta = 0$$

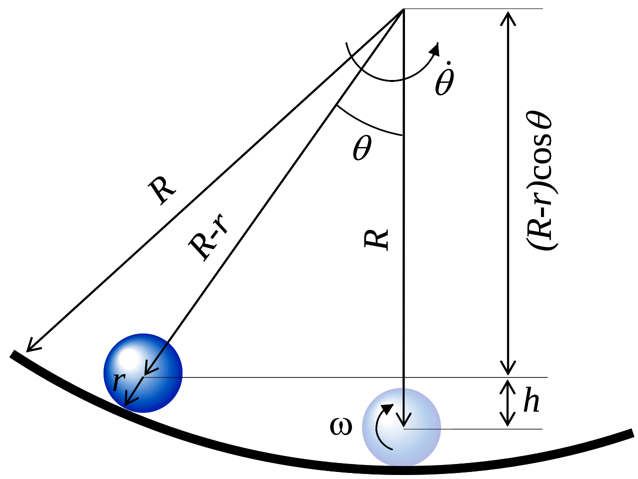
assuming for small angles $\sin\theta \approx \theta$ and so the natural frequency is

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.81}{0.7}} = 3.74\,\text{Hz}$$

Example 12.2

The natural period of a building or structure can be lengthened to alleviate the highest earthquake forces using isolating bearings that consist of a ball carrying the structural weight and rolling on a concave surface (e.g. www.earthquakeprotection.com). For a bridge, sixteen such bearings are to be used in which the high strength steel ball has a radius of 10cm and the concave bearing surface a radius of curvature of 1.5m, find the natural frequency of a bearing.

If the bearing is shaken by an earthquake with a seismic frequency of 40Hz that produces sinusoidal motion of the ground of 150mm what will be the form of the equation of motion of the bearing.



Solution:

Kinetic energy at bottom of bearing:

$$KE = \frac{mv^2}{2} + \frac{I\omega^2}{2}$$

and $\dot{\theta} = \dfrac{v}{R-r} = \dfrac{\omega\, r}{R-r}$ also $I = \dfrac{mr^2}{2}$

so substituting for v, ω and I:

$$KE = \frac{3m(R-r)^2\dot{\theta}^2}{4}$$

The potential energy gained when the ball is displaced from bottom of the bearing is:

$$PE = mgh = mg(R-r)\,(1-\cos\theta)$$

now $(1-\cos\theta) = 2\sin^2\dfrac{\theta}{2} \approx \dfrac{\theta^2}{2}$ for small angles, so $PE = mg(R-r)\dfrac{\theta^2}{2}$

Thus the energy equation is:

$$KE + PE = \frac{3m(R-r)^2\dot{\theta}^2}{4} - gm(R-r)\frac{\theta^2}{2}$$

And differentiating with respect to time:

$$\frac{3(R-r)^2\dot{\theta}\ddot{\theta}}{2} - g(R-r)\theta\dot{\theta} = 0 \quad \text{or} \quad (R-r)\,\left(3(R-r)\,\ddot{\theta} - 2g\theta\right)\,\dot{\theta} = 0$$

Since $\dot{\theta}$ is not always zero then:

$$\ddot{\theta} - \frac{2g}{3(R-r)}\theta = 0$$

And the natural frequency is:

$$\omega_n = \sqrt{\frac{2g}{3(R-r)}} = \sqrt{\frac{2\times 9.81}{3(1.5-0.1)}} = 2.16\ \text{Hz}$$

The earthquake will generate a forcing term of the form $F_0 \sin\omega\, t$ so that the equation of motion will be $\ddot{\theta} - \dfrac{2g}{3(R-r)}\theta = F_0 \sin 80\pi\ t$.