EVERYDAY EXAMPLES OF ENGINEERING CONCEPTS

D11: 3D kinetics of rigid bodies

This is an extract from 'Real Life Examples in Dynamics: Lesson plans and solutions' edited by Eann A. Patterson, first published in 2006 (ISBN:978-0-615-20394-2) which can be obtained on-line at www.engineeringexamples.org and contains suggested exemplars within lesson plans for Sophomore Solids Courses. Prepared as part of the NSF-supported project (#0431756) entitled: "Enhancing Diversity in the Undergraduate Mechanical Engineering Population through Curriculum Change".

INTRODUCTION

(from *'Real Life Examples in Dynamics: Lesson plans and solutions'*)

These notes are designed to enhance the teaching of a junior level course in dynamics, increase the accessibility of the principles, and raise the appeal of the subject to students from diverse backgrounds. The notes have been prepared as skeletal lesson plans using the principle of the 5Es: Engage, Explore, Explain, Elaborate and Evaluate. The 5E outline is not original and was developed by the Biological Sciences Curriculum Study[1] in the 1980s from work by Atkin and Karplus[2] in 1962. Today this approach is considered to form part of the constructivist learning theory and a number of websites provide easy-to-follow explanations of them[3].

These notes are intended to be used by instructors and are written in a style that addresses the instructor, however this is not intended to exclude students who should find the notes and examples interesting, stimulating and hopefully illuminating, particularly when their instructor is not utilizing them. In the interest of brevity and clarity of presentation, standard derivations and definitions are not included since these are readily available in textbooks which these notes are not intended to replace but rather to supplement and enhance. Similarly, it is anticipated that these lessons plans can be used to generate lectures/lessons that supplement those covering the fundamentals of each topic.

It is assumed that students have acquired a knowledge and understanding of topics usually found in a Sophomore level course in Statics, including free-body diagrams and efficiency.

This is the second in a series of such notes. The first in the series entitled 'Real Life Examples in Mechanics of Solids' edited by Eann Patterson (ISBN: 978-0-615-20394-2) was produced in 2006 and is available on-line at www.engineeringexamples.org.

Acknowledgements

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[1] Engleman, Laura (ed.), *The BSCS Story: A History of the Biological Sciences Curriculum Study*. Colorado Springs: BSCS, 2001.
[2] Atkin, J. M. and Karplus, R. (1962). Discovery or invention? *Science Teacher* 29(5): 45.
[3] e.g. Trowbridge, L.W., Bybee, R.W., *Becoming a secondary school science teacher*. Merrill Pub. Co. Inc., 1990.

THREE-DIMENSIONAL RIGID BODY MOTION

11. Topic**: Kinetics of rigid bodies in three dimensions**

Engage:

Take a child's spinning top into class. In fact wood ones are cheap enough for you to take a handful into class so that you can pass them around in case the students have forgotten what they are like. Encourage the students to spin the tops. Also take the front wheel of your bicycle.

Explore:

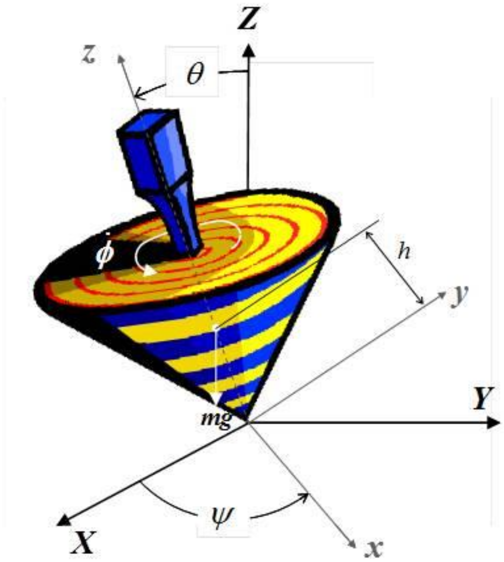
Spin your top sufficiently fast that its spin axis remains vertical – this is known as sleeping. Of course, friction will reduce the spin rate, $\dot{\phi}$, so that the top starts to lean over which is known as nutation, and the axis about which it is spinning will rotate about the vertical axis which is known as precession.

In practice, the spin rate of the top will continue to decrease due to friction but in steady precession it is assumed that the rate of spin is constant.

Explain :

Discuss the concept of two co-ordinate systems: one fixed to the bench, *XYZ*; and one fixed to the top, *xyz*; but with a common origin at the point of the top.

Re-iterate that three angles are required to define the three-dimensional motion of a rigid body relative to an axis: one angle to define the rotation of the body about the axis and two angles to define the orientation of the axis. We just defined these for the top, i.e. the spin angle, ϕ, plus the nutation, θ and precession angles, ψ. These are known as Euler's angles and the equations of motions can be expressed in terms of them, i.e.



$$\sum M_x = I_{xx}\ddot{\theta} + \left(I_{zz} - I_{xx}\right)\,\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta$$

$$\sum M_y = I_{xx}\left(\ddot{\psi}\sin\theta + 2\dot{\psi}\dot{\theta}\cos\theta\right) - I_{zz}\left(\dot{\phi}\dot{\theta} + \dot{\psi}\dot{\theta}\cos\theta\right)$$

$$\sum M_z = I_{zz}\left(\ddot{\phi} + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta\right)$$

Generally these equations have to be solved numerically; however when the precession rate is constant these equations reduce to:

$$\sum M_x = \left(I_{zz} - I_{xx}\right)\,\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta\,,\quad \sum M_y = 0 \text{ and } \sum M_z = 0\,.$$

Now, the mass of the top exerts a moment about the origin of: $\sum M_x = mgh\sin\theta$

and hence,

$$mgh\sin\theta = \left(I_{zz} - I_{xx}\right)\,\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta$$

or

$$\dot{\phi} = \frac{mgh\sin\theta - \left(I_{zz} - I_{xx}\right)\,\dot{\psi}^2 \sin\theta\cos\theta}{I_{zz}\dot{\psi}\sin\theta}$$

So, if you estimate the nutation (lean), θ and the precession rate, ψ which is usually fairly slow then you can calculate the spinning rate, $\dot{\phi}$.

For the cone in the diagram above, if it is of radius and length 6cm and made of teak (density, ρ = 700 kg/m^3) then:

$$m = \rho V = \rho\frac{\pi R^3 h}{3} = 700\times\frac{\pi\times 0.06^4}{3} = 0.0095\,\text{kg}$$

and

$$I_{xx} = I_{yy} = m\left(\frac{3h^2}{5} + \frac{3R^2}{20}\right) = 0.0095\times\left(\frac{3\times 0.06^2}{5} + \frac{3\times 0.06^2}{20}\right) = 2.57\times 10^{-5}\,\text{kg.m}^2$$

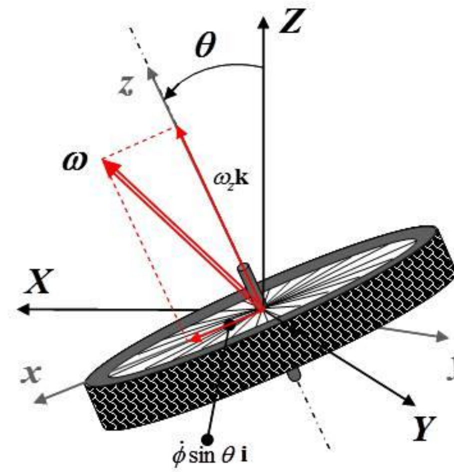
$$I_{zz} = \frac{3mR^2}{10} = \frac{3\times 0.0095\times 0.06^2}{10} = 1.03\times 10^{-5}\,\text{kg.m}^2$$

Now, estimating the nutation to be about 15° and the precession rate to be about ten revolution per minute (= 1.05 rads/s):

$$\dot{\phi} = \frac{0.0095\times 9.81\times\left(\tfrac{4}{3}\times 0.06\right)\sin 15 - \left(1.03 - 2.56\right)\times 10^{-5}\times 1.05^2\sin 15\cos 15}{1\times 10^{-5}\times 1.05\sin 15} = 712\,\text{rad/s}$$

Or about 113 revolutions/second which 6800 r.p.m.

Elaborate



Engage the students by spinning your bicycle wheel and holding by the axle stubs in front of you so the axle is horizontal. Tilt the axle by raising one end of the axle and lower the other end so that you experience gyroscopic moments restoring the axle to its original position. Explain to the students about the forces you are experiencing. Perhaps have a student come to the front of the class and hold the wheel while you spin it for them.

The angular velocity of the wheel is the vector sum of rate of spin about the local z-axis, ϕ; nutation about the global Y-axis, θ and precession about the global Z-axis, ψ:

$$\boldsymbol{\omega} = \dot{\psi}\,\mathbf{K} + \dot{\theta}\,\mathbf{j} + \dot{\phi}\,\mathbf{k}$$

where **i**, **j**, and **k** are unit vectors along the local axes and **K** is the unit vector along the fixed Z axis, such that $\mathbf{K} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k}$

hence, the angular velocity is given by: $\boldsymbol{\omega} = -\dot{\psi}\sin\theta\,\mathbf{i} + \dot{\theta}\,\mathbf{j} + \left(\dot{\phi} + \dot{\psi}\cos\theta\right)\mathbf{k}$

The local axes of rotation, *xyz* are also the axes of inertia, so the angular momentum, $\mathbf{H_o}$ will be the sum of the moments of inertia multiplied by the appropriate component of the angular velocity, i.e.

$$\mathbf{H}_0 = -I'\dot{\psi}\sin\theta\,\mathbf{i} + I'\dot{\theta}\,\mathbf{j} + I\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\mathbf{k}$$

where I and I' are the moments of inertia about the spin axis and fixed axes respectively. Newton's second law of motion in terms of moments is:

$$\sum \mathbf{M}_O = \dot{\mathbf{H}}_O$$

and it can be shown that:

$$\sum \mathbf{M}_O = \left(\dot{\mathbf{H}}_O\right)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_O$$

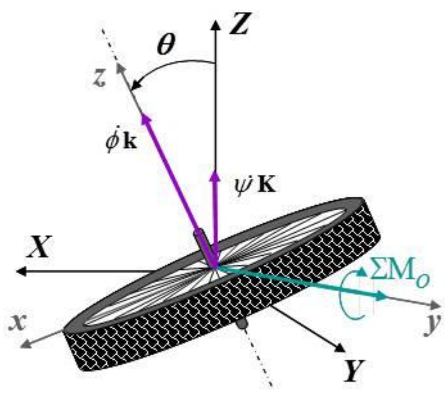
where $\left(\dot{\mathbf{H}}_O\right)_{xyz}$ is the rate of change of $\mathbf{H}_O$ with respect to rotating axes *xyz* and $\boldsymbol{\Omega}$ is the angular velocity of the rotating axes *xyz*. Now, these axes nutate and precess but do not spin thus,

$$\boldsymbol{\Omega} = -\dot{\psi}\sin\theta\,\mathrm{i} + \dot{\psi}\cos\theta\,\mathrm{k}$$

Now consider the case where, the nutation, θ, rate of precession, $\dot{\psi}$ and rate of spin, $\dot{\phi}$ are all constant then the angular momentum, $\mathbf{H}_O$ is also constant, i.e. $\left(\dot{\mathbf{H}}_O\right)_{xyz} = 0$ then

$$\sum \mathbf{M}_O = \boldsymbol{\Omega} \times \mathbf{H}_O$$

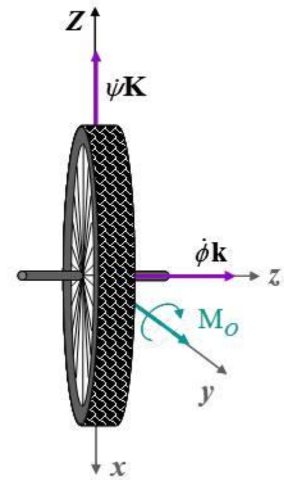
$$\sum \mathbf{M}_O = \left[I\left(\dot{\phi} + \dot{\psi}\cos\theta\right) - I'\dot{\psi}\cos\theta\right]\dot{\psi}\sin\theta\,\mathbf{j}$$



This is the couple that must be applied to maintain the steady precession, i.e. $\dot{\theta} = \ddot{\psi} = \ddot{\phi} = 0$ assuming that the center of gravity of the wheel is fixed in space (ΣF=0).

Now, for the situation in the demonstration the precision and spin axes are at right angles, i.e. $\theta = 90°$ so

$$\sum \mathbf{M}_O = I\dot{\phi}\dot{\theta}\,\mathbf{j}$$



When this is applied about an axis perpendicular to the axis of spin the wheel will precess about an axis perpendicular to both the axis of spin and the couple, i.e. the global *Z*-axis.

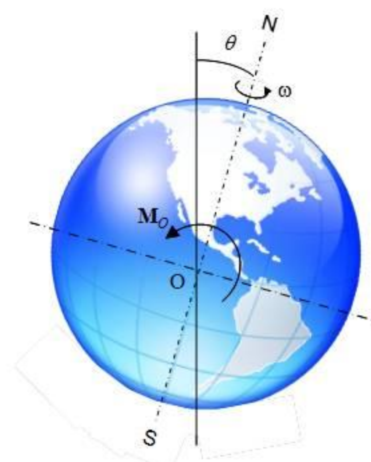
This behavior makes it easier to balance on your bicycle at higher speeds due to the stabilizing effects of the spinning wheels. It is also known as gyroscope motion.

Evaluate

Ask students to attempt the following examples:

Example 11.1

The Earth can be thought of as a spinning top with a spin axis through the North and South Poles which slowly rotates, or precesses so that the North Pole draws out a circle in space. This precession is very slow (one degree every 71.6 years) and is known as the Precession of the Equinoxes. It was probably observed first by Hipparchus in around 137BC and results in a slow change in the position of the sun with respect to the stars at an equinox. This is important for calendars and their leap year rules.



The Precession of the Equinoxes is caused by a (force) couple acting on the earth due to the gravitational attraction of the sun and moon. Assuming that the earth is an oblate spheroid of average radius, 3960 miles and mass 5.9742×10^{24}kg, and that the earth's nutation is constant at 23.45°, calculate the couple.

Solution

Rate of precession, $\dot{\psi} = \dfrac{1}{71.6}\times\dfrac{2\pi}{360}\times\dfrac{1}{365\times24\times3600} = 7.73\times10^{-12}\,\text{rad/s}$

Rate of spin, $\dot{\phi} = \dfrac{2\pi}{24\times3600} = 7.27\times10^{-5}\,\text{rad/s}$

Assume a moment of inertia for a sphere,

$$I = \frac{2mR^s}{5} = \frac{2 \times 5.9742 \times 10^{24} \times \left(3960 \times \frac{8000}{5}\right)^2}{5} = 9.59 \times 10^{37}\,\text{kg.m}^2$$

And $\sum \mathbf{M}_O = \mathbf{\Omega} \times \mathbf{H}_O = \left(I\left(\dot{\phi} + \dot{\psi}\cos\theta\right) - I'\dot{\psi}\cos\theta\right)\dot{\psi}\sin\theta\,\mathbf{j}$

$$\sum \mathbf{M}_O = \left(I\dot{\phi} + \left(I - I'\right)\dot{\psi}\cos\theta\right)\dot{\psi}\sin\theta\,\mathbf{j}$$

assuming $I \approx I'$ then:

$$\mathbf{M}_O = I\dot{\phi}\dot{\psi}\sin\theta = \left(9.59 \times 10^{37}\right) \times \left(7.27 \times 10^{-5}\right) \times \left(7.73 \times 10^{-12}\right)\sin 23.45$$

and finally:

$$\mathbf{M}_O = 2.14 \times 10^{22}\,\text{Nm}$$

Example 11.2

Ask the students to design a gyroscope stabilizer for an experienced rider on a motorcycle.